UDC 532.5

NOTES ON THE MATHEMATICAL THEORY OF SMALL OSCILLATIONS OF AN INVISCID GAS

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The problem of completeness of a system of natural oscillations for an inviscid gas in proper or external gravity field is discussed. It is shown that such completeness occurs for small (adiabatic) oscillations of a gravitating gaseous sphere about its position of mechanical equilibrium. The proof (which can be used in the cases of separation of variables) is reduced to the general theorem of completeness for self-conjugate operators.

1. Let an inviscid gas filling a region G be maintained in the state of mechanical equilibrium by a gravitational field. Then

$$\frac{1}{\rho_0}\operatorname{grad} p_0 + \operatorname{grad} V_0 = 0 \tag{1.1}$$
$$V_0(\mathbf{x}) = -\gamma \int \frac{\rho_0(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y} + \Phi(\mathbf{x}), \quad \mathbf{x} = (x^1, x^2, x^3)$$

where γ is the gravitational constant and $\Phi(\mathbf{x})$ is the potential of the external field. On the free boundaries of G we have $p_0 = 0$ and $\rho_0 = 0$. The hydrodynamic equations linearized near the state of equilibrium are

$$\partial \mathbf{u} / \partial t = -(1/\rho_0)\operatorname{grad} p_1 + (\rho_1/\rho_0^2)\operatorname{grad} p_0 - \operatorname{grad} V_1$$
$$\frac{\partial \rho_1}{\partial t} = -\operatorname{div}(\rho_0 \mathbf{u}), \qquad \frac{\partial s_1}{\partial t} = -u^k \frac{\partial s_0}{\partial x^k} \tag{1.2}$$
$$p_0 = p(\rho_0, s_0), \qquad p_1 = \left(\frac{\partial p}{\partial \rho}\right)_0 \rho_1 + \left(\frac{\partial p}{\partial s}\right)_0 s_1$$
$$V_1(\mathbf{x}) = -\gamma \int \frac{\rho_1(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y}$$

Differentiating (1.2) with respect to t and eliminating ρ_1 and s_1, we obtain

$$\partial^2 \mathbf{u} / \partial t^2 = -L\mathbf{u}$$
$$L\mathbf{u} = -\frac{1}{\rho_0}[\operatorname{grad}(a \operatorname{div}\mathbf{u}) + \operatorname{grad}(\mathbf{b}\cdot\mathbf{u})] + \frac{1}{\rho_0}\mathbf{b}\operatorname{div}\mathbf{u} +$$
$$\frac{1}{\rho_0}Q\mathbf{u} + \operatorname{grad} V_1^{\cdot} \equiv D\mathbf{u} + \operatorname{grad} V_1^{\cdot} \tag{1.3}$$

where

$$a = \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_0, \quad \mathbf{b} = \operatorname{grad} p_0, \quad Q_{jk} = \frac{\partial p_0}{\partial x^j}\frac{\partial \rho_0}{\partial x^k}\frac{1}{\rho_0} \tag{1.4}$$
$$V_1^{\cdot} = \gamma \int \frac{\operatorname{div}(\rho_0(\mathbf{y})\,\mathbf{u}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y} \tag{1.5}$$

It is easy to verify that the operator L is formally self-conjugate (symmetric) in the sense of the following scalar product:

$$(\mathbf{u}\cdot\mathbf{v})_\rho = \int \rho_0(\mathbf{x}) \sum_{k=1}^{3} u^k(\mathbf{x})\,\bar{v}^k(\mathbf{x})\,d\mathbf{x} \tag{1.6}$$

To be able to utilize the general theorem on completeness we must prove the real self-conjugacy, i.e. we must show that the domains of definition of L and L^* coincide.

The difficulties which are experienced are connected with the strong degeneracy of the differential operator D (see (1.3)). In particular, the characteristic determinant of the major terms of D is equal to zero. Degeneration to the system $Du = \lambda u$ makes the general theorems on elliptic equations inapplicable (this is what makes this problem interesting from the mathematical point of view). In addition to D (1.3) also contains the integral term V_1'. We shall show that this complication does not affect the feasibility of application of the general theorem on completeness of eigenfunctions.

In fact, let H_ρ be a space of functions $\mathbf{u}(\mathbf{x})$ such that

$$\|\mathbf{u}\|_\rho^2 = \int \rho_0(\mathbf{x})\,|\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} < \infty$$

with the scalar product (1.6). Let

$$\Gamma\mathbf{u} = \operatorname{grad}\left[\int \frac{1}{|\mathbf{x}-\mathbf{y}|}\operatorname{div}(\rho_0(\mathbf{y})\,\mathbf{u}(\mathbf{y}))\,d\mathbf{y}\right] \tag{1.7}$$

For the smooth function $\mathbf{u}(\mathbf{x})$ we rewrite (1.7) as follows:

$$\Gamma\mathbf{u} = \int I(\mathbf{x}-\mathbf{y})\,\rho_0(\mathbf{y})\,u(\mathbf{y})\,d\mathbf{y} \tag{1.8}$$

where the matrix

$$I = (I_{jk}) = \left(\frac{\partial^2}{\partial x^j \partial x^k}\,\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \tag{1.9}$$

The operator Γ written in the form (1.8) is bounded, i.e. is defined everywhere in H_ρ. Indeed, if $\mathbf{u} \in H_\rho$, then $\rho_0\,\mathbf{u} \in H(G)$, where $H(G)$ is a Hilbert space with the usual norm

$$\|\mathbf{u}\|^2 = \int_G |\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x}$$

The kernel I generates a bounded operator in $H(G)$ (see [1]), therefore we have

$$\|\Gamma\mathbf{u}\|_\rho \leqslant C\,\|\Gamma\mathbf{u}\| \leqslant C_1\,\|\rho_0\mathbf{u}\| \leqslant C_2\,\|\mathbf{u}\|_\rho$$

We thus see that in investigating the domain of definition of the (unbounded) operator $L = D + \gamma\Gamma$, we can consider its differential part D (see (1.3)).

In contrast to the classical oscillation problems, Eqs. (1.3) have an aboundant family of null eigenfunctions. Such are all $\mathbf{u}(\mathbf{x})$ for which

$$\operatorname{div}(\rho\,\mathbf{u}) = 0, \qquad u_n = \left(\mathbf{u},\ \frac{\operatorname{grad}\rho_0}{|\operatorname{grad}\rho_0|}\right) = 0 \tag{1.10}$$

Each of these $\mathbf{u}(\mathbf{x})$ has a corresponding motion along an equipotential surface ($V_0(\mathbf{x}) =$ const) for which $p_1(\mathbf{x}) \equiv 0$. Let H_ρ^+ be an orthogonal complement of the functions (1.10) in H_ρ. The problem consists of proving the completeness in H_ρ^+ of the remaining "nontrivial" eigenfunctions.

2. Let us now consider a gaseous sphere of radius R ($\rho_0 = \rho_0(r) \equiv 0$ when $r \geqslant R$). We assume that: (1) in some neighborhood of the boundary $r > R - \delta$ the adiabatic index is constant

$$(\partial \ln p\,/\,\partial \ln \rho)_s = \varkappa \tag{2.1}$$

and, (2) on approaching the boundary ρ_0, it rapidly tends to zero

$$\rho_0(r) = C((\Delta r)^\nu + o(\Delta r)^\nu), \qquad \nu > 1, \quad \Delta r = R - r \tag{2.2}$$

Here we naturally assume a certain regularity in behavior of ρ_0 and p_0 when $r \to R$ (see notes Sect. 5). We note that (2.2) always holds, provided that the equilibrium is stable and $\varkappa < 2$. In fact, for the convection not to appear, it is necessary that the entropy $s_0(r)$ does not decrease with increasing r (see e.g. [2], Sect. 4). Therefore for $r > R - \delta$.

$$p_0 \geqslant C\rho_0^\varkappa$$

On the other hand (see (1.1))

$$\frac{1}{\rho_0} p_0' = -V_0', \qquad V_0'(R) \neq 0 \tag{2.3}$$

consequently

$$p_0(r) \leqslant C_1 \ (\Delta r)^{\nu+1} \rho_0(r) \leqslant C_2(\Delta r)^\nu, \quad \nu = \frac{1}{\varkappa - 1}$$

On separating the variables we obtain a system with a singularity for $r = R$ and a problem of asymptotic boundary conditions with $r \to R$ arises.

We find that formally, within the framework of the theory of self-conjugate operators, no boundary condition is required at all. To be more precise, when (2.2) holds, the boundary condition is satisfied by the requirement that $\mathbf{u} \in H_\rho$. Section 3 deals with the proof of this assertion.

3. For a sphere the conditions (1.10) assume the form $u_r = 0$, $\operatorname{div} \mathbf{u} = 0$. In the space $H_\rho{}^+$, u_r is arbitrary and the following condition holds:

$$\frac{\partial u_\theta}{\partial \varphi} - \frac{\partial}{\partial \theta}(u_\varphi \sin\theta) = 0$$

(More accurately, $H_\rho{}^+$ is the closure of the set of smooth functions satisfying this condition).

We set

$$u_r(r, \theta, \varphi) = w_1(r) Y_{lm}(\theta, \varphi), \quad u_\theta = w_2 \frac{\partial Y_{lm}}{\partial \theta}, \quad u_\varphi = \frac{w_2}{\sin\theta} \frac{\partial Y_{lm}}{\partial \varphi} \tag{3.1}$$

Then

$$\operatorname{div} \mathbf{u} = zY_{lm}, \quad z(r) = \frac{dw_1}{dr} + \frac{2}{r} w_1 - \frac{l(l+1)}{r} w_2 \tag{3.2}$$

The operator $L = D + \gamma\Gamma$ (see (1.3) and (1.7)) generates on the functions (3.1) a one-dimensional integro-differential operator

$$L^{(l)} = D^{(l)} + \gamma\Gamma^{(l)}$$

$$D^l \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\rho_0} \dfrac{d}{dr}(az + bw_1) + \dfrac{b}{\rho_0} z + \dfrac{q}{\rho_0} w_1 \\ -\dfrac{1}{\rho_0 r}(az + bw_1) \end{pmatrix} \tag{3.3}$$

where (see (1.4))

$$a = a(r) = \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_0, \quad b = p_0'(r), \quad q = \frac{p_0' \rho_0'}{\rho_0} \tag{3.4}$$

The following expression corresponds to the scalar product (1.6):

$$(\mathbf{w} \cdot \boldsymbol{\omega}) = (\mathbf{w} \cdot \boldsymbol{\omega})_\rho{}^l = \int_0^R \rho_0(r) r^2 [w_1\bar{\omega}_1 + l(l+1) w_2\bar{\omega}_2] \, dr \tag{3.5}$$

We denote the space $\mathbf{w} = (w_1, w_2)$ with the scalar product (3.5) by $H_\rho^{(l)}$ (when $l = 0$ it is necessary to consider the space $H_\rho^{(0)}$ of the scalar functions $w_1(r)$, and the first line of (3.3) only. The operator $\Gamma^{(l)}$ is bounded self-conjugate in $H_\rho^{(l)}$, and the operator $D^{(l)}$ is formally self-conjugate in the sense of (3.5), i.e.

$$(D^{(l)}\mathbf{w}\cdot\boldsymbol{\omega}) - (\mathbf{w}\cdot D^{(l)}\boldsymbol{\omega}) = \lim_{\varepsilon\to 0} r^2 a(r)(z\bar{\omega}_1 - \bar{\zeta}w_1)\Big|_{r=\varepsilon}^{r=R-\varepsilon} \quad (3.6)$$

$$\left(\zeta = \frac{d\omega_1}{dr} + \frac{2}{r}\omega_1 - \frac{l(l+1)}{r}\omega_2\right)$$

The largest family of functions on which $D^{(l)}$ can be considered consists of all $\mathbf{w} \in H_\rho^{(l)}$, for which $D^{(l)}\mathbf{w} \in H_\rho^{(l)}$. We shall show that in this largest region $D^{(l)}$ (and therefore $L^{(l)}$) is symmetric, i.e. the right-hand side of (3.6) is equal to zero. In accordance with the general principles (see [3]) it is sufficient to prove that no solution $D^{(l)}\mathbf{w} = i\mathbf{w}$ exists such that $\mathbf{w} \in H_\rho^{(l)}$. If such a function exists, then (z is given by (3.2))

$$(D^{(l)}\bar{\mathbf{w}}\cdot\mathbf{w}) = \lim_{\varepsilon_1,\varepsilon_2\to 0}\left\{\int_{\varepsilon_1}^{R-\varepsilon_2}[a|z|^2 + b(z\bar{w}_1 + \bar{z}w_1) + q|w_1|^2]r^2dr + r^2a(r)z\bar{w}_1\Big|_{\varepsilon_1}^{R-\varepsilon_2}\right\} = i\int_0^R(|w_1|^2 + l(l+1)|w_2|^2)r^2dr \quad (3.7)$$

The integral in the left-hand side is real, and a proof that the boundary terms vanish when $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ will lead to contradiction. The system $D^{(l)}\mathbf{w} = \lambda\mathbf{w}$ can be reduced, for any fixed $\lambda \neq 0$, to the form

$$\frac{d\mathbf{w}}{dr} = A(r)\mathbf{w}, \quad A(r) = \frac{A_{-1}}{R-r} + B(r) \quad (3.8)$$

($B(r)$ is a smooth function in the neighborhood of $r = R$.) The computations yield the following results for the eigenvalues A_{-1}: $\mu_1 = 0$ and $\mu_2 = -\nu$, where ν is the power index given in (2.2). Therefore we have two particular solutions

$$\mathbf{w}^{(1)}(r) = C^{(1)} + o(1), \quad \mathbf{w}^{(2)}(r) = (\Delta r)^{-\nu}(C^{(2)} + o(1)) \quad (3.9)$$

If $\nu > 1$, from the condition $D^{(l)}\mathbf{w} = i\mathbf{w}$, $\mathbf{w} \in H_\rho^{(l)}$ it follows that $w_1(r) = O(1)$ when $r \to R$. Then from (3.8) we have $|w_1'| \leqslant C/\Delta r$ and

$$|z| = \left|w_1' + \frac{2}{r}w_1 - \frac{l(l+1)}{r}w_2\right| \leqslant \frac{C_1}{\Delta r}$$

It remains to note that $a = \varkappa p_0$ when $r > R - \delta$ and, that $p_0(r) = O((\Delta r)^{\nu+1})$ by virtue of (2.2) and (2.3). Therefore $r^2 a(r) z(r)\bar{w}_1(r) \to 0$ when $r \to R$. Similarly we can prove that the boundary terms in (3.7) (and therefore in (3.6)) tend to zero as $\varepsilon_1 \to 0$.

Thus for $D^{(l)}$ defined on its maximum domain of definition, $D^{(l)} = (D^{(l)})^*$ Since $\Gamma^{(l)}$ is bounded on the same domain, $L^{(l)} = (L^{(l)})^*$ and the general theorem on the completeness of eigenfunctions is applicable.

Since only the oscillations of a limited volume of gas are considered, it would be reasonable to expect a discrete spectrum of eigenvalues. We should however stress that the property of discreteness does not follow from any standard theorems since the boundary is singular (for $r = R$) and must be proved separately.

The system $D^{(l)}\mathbf{w} = \lambda\mathbf{w}$ has one second-order equation and one first-order equation. It is however degenerate; when $\lambda \neq 0$ it reduces to (3.8), while when $\lambda = 0$, it has a unique solution $\mathbf{w} = 0$. It is the degeneracy that causes the eigenvalues of $D^{(l)}$ (and $L^{(l)}$) to have not only ∞ but also zero as their limiting points. This has been apparent for a long time to physicists investigating the spectrum of this problem (see e. g. [4]).

4. Setting together everything that has been said in Sects. 1 – 3, we obtain the following theorem.

Theorem. Let a gaseous sphere of radius R be in the state of equilibrium in its own gravitational field, and let

$$\rho_0(r) \sim C(R-r)^{\nu}, \quad \nu > 1 \text{ when } r \to R \tag{4.1}$$

Then any vector function $\mathbf{u}(\mathbf{x})$ for which

$$\int \rho_0(\mathbf{x})\,|\,\mathbf{u}(\mathbf{x})\,|^2\,d\mathbf{x} < \infty \tag{4.2}$$

can be expanded into a series of the form

$$\mathbf{u}(\mathbf{x}) = (u_r, u_\theta, u_\varphi) = \psi_0(\mathbf{x}) + \sum_{l,m,n} c_{lmn}\psi_{lmn}(\mathbf{x}) \tag{4.3}$$

Here $\psi_{lmn}(\mathbf{x})\, e^{i\omega_{ln}t}$ and $\psi_0(\mathbf{x})$ are solutions of the equation (1.3) describing small oscillations and written for the velocity components. The behavior of $\rho_0(r)$ in the condition (4.1) must be sufficiently regular (see Sect. 5).

Notes. The family $\psi_0(x)$ depends on a single arbitrary function. The smooth functions $\psi_0(x)$ have the form

$$\psi_0(x) = \left(0, \frac{1}{\sin\theta}\frac{\partial g}{\partial\varphi}, -\frac{\partial g}{\partial\theta}\right), \qquad g = g(r, \theta, \varphi)$$

and the remaining ψ_0 represent the limits of the above functions in the sense of the metric (4.2). The nontrivial eigenfunctions have the form (see Sect. 3)

$$\psi_{lmn}(x) = \left(w_1(r)Y_{lm}(\theta, \varphi), \quad w_2(r)\frac{\partial Y_{lm}}{\partial\theta}, \frac{w_2(r)}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}\right)$$

$$(0 \leqslant m \leqslant l, \ 0 \leqslant l < \infty, \ 0 \leqslant n < \infty)$$

and the series (4.3) converges, generally speaking, in the mean (in the sense of the metric (4.2)).

5. Notes. (Re Sect. 1). The systems (1.2) and (1.3) are not equivalent. The system (1.2) has nonzero stationary solutions for which $\mathbf{u}(\mathbf{x}) \equiv 0$. (These solutions arise because of the presence of the similarity transformations which change one solution of (1.1) into another). They have a corresponding trivial solution of (1.3) $\mathbf{u}(\mathbf{x}) \equiv 0$. Conversely however, if we assume that $L\varphi(\mathbf{x}) = 0$, then $\mathbf{u}(\mathbf{x}, t) = \varphi(x)\,t$ is a solution of (1.3), but there is no corresponding expression in (1.2). Finally, there is a reciprocal one-to-one correspondence between the eigenfunctions of (1.2) and (1.3) possessing nonzero eigenvalues.

(Re Sect. 2). Strictly speaking, the reference to standard theorems of the theory of differential (nondegenerate) operators is not valid. Carrying out all the proofs for the degenerate case under consideration we obtain an exact description of the domain of definition $\Omega(D^{(l)})$. Namely, $w \in \Omega(D^{(l)})$ if: (i) w_1' and z' exist and are locally square integrable and (ii) the integrals (for z see (3.2))

$$\int_0^R \frac{1}{\rho_0}[az + bw_1]^2\,dr, \qquad \int_0^R \frac{r^2}{\rho_0}\left[\frac{d}{dr}(az + bw_1) + bz + qw_1\right]^2 dr$$

converge.

We note that the derivative w_2' need not exist. The proofs are fairly straightforward (see [2]) and are therefore not given here.

Another method of overcoming the difficulties connected with the degeneracy is suitable for the case of an external field. Here the system $L^{(l)} w = \lambda \mathbf{w}$ is reduced to a single equation and highly special theorems on completeness are used. This method is used in [5].

(Re Sect. 3). The eigenvalues A_{-1} are used to judge the behavior of the complete system (3.8). For (3.9) to hold it is sufficient that

$$\int^{R} |B_{jk}(r)|\, dr < \infty, \qquad j, k = 1, 2 \tag{5.1}$$

Let us write out the elements of the matrix $A = A(r, \lambda)$

$$A_{11} = \frac{2}{r} + \frac{b}{a}, \quad A_{12} = \frac{\lambda\rho_0 r}{a} - \frac{l(l+1)}{r}$$

$$A_{21} = \frac{1}{\lambda\rho_0 r}\left(q - \frac{b^2}{a} - \lambda\rho_0\right), \quad A_{22} = -\frac{b}{a} + \frac{(\rho_0 r)'}{\rho_0 r}$$

Expressions (5.1) are satisfied if (with $\varkappa = \mathrm{const}$)

$$\frac{b}{a} = \frac{\nu+1}{\varkappa\Delta r} + O((\Delta r)^{-\alpha}), \qquad \frac{\rho_0'}{\rho_0} = \frac{\nu}{\Delta r} + O((\Delta r)^{-\alpha}), \quad \alpha < 1$$

In particular, when $r = R$, it is sufficient for the functions p_0 / p' and ρ_0 / ρ_0' to be differentiable.

(Re Sects. 2 and 3). The concept of a "physical" boundary condition at the free boundary was not used at all. It is therefore expedient to explain the connection between the conditions that $\mathbf{u}$, $L\mathbf{u} \in H_\rho$ which are utilized here and the requirement that $p \to 0$ as $r \to R$. The formulas of Sects. 2 and 3 yield the following assertion.

Let the following be true for the solution of (1.2)

$$\int \rho_0 |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \qquad \int \rho_0 |\mathbf{u}_{tt}(\mathbf{x}, t)|^2 d\mathbf{x} < \infty$$

and let (2.2) hold. Then every term of the expansion $\partial p_1 / \partial t = \Sigma \Pi^{lm}(r, t) Y_{lm}(\theta, \varphi)$ tends to zero as $r \to R$.

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